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EQUATIONS OF GEODESICS IN A TWO-DIMENSIONAL FINSLER SPACE WITH SPECIAL (α, β) METRIC

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INTRODUCTION :

. The geodesic equation in a two-dimensional Finsler space is given by the differential equation of the Weierstrass form. In the year 2000 Matsumoto and Park express the differential equations of geodesics in a two dimensional Finsler space with a generalized Kropina metric. The purpose of present paper, we express the differential equations of geodesics in a two-dimensional Finsler space with special (α, β) metric is of the form

$L = \alpha + \beta + \frac{\beta^2}{\alpha - \beta}$.

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1. Introduction:-

In the year 1997 Matsumoto and Park [1] obtained the equation of geodesics in two dimensional Finsler spaces with the Randers metric $(L = \alpha + \beta)$and the Kropina metric$L = (\alpha^2/\beta)$, whereas in 1998, they have [2] obtained the equation of geodesic in two-dimensional Finsler space with the slope metrics, i.e. Matsumoto metric given by $L = \alpha^2/(\alpha - \beta)$, by considering β as an infinitesimal of degree one and neglecting infinitesimal of degree more than two they obtained the equations of geodesic of two-dimensional Finsler space in the form $y'' = f = (x, y, y')$, where (x, y) are the co-ordinate of two-dimensional Finsler space.

The study on the differential equations of geodesics in a two-dimensional Finsler space$F^2 = (M^2, L)$with an (α, β)-metric is interesting and useful. The geodesics of F^2 are regarded as the curves of an associated Riemannian space $R^2 = (M^2, \alpha)$ which are bent by the differentia 1-form β. Recently, M. Matsumoto and the first author ([9]) have expressed the differential equations of the geodesics in two-dimensional Randers spaces and Kropina spaces in the clearest form$y'' = f(x, y, y')$.

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The purpose of the present paper is devoted to studying the differential equations of geodesics in a two-dimensional Finsler space with special (α,β)metric.

2.Preliminaries

Let $F^2=(M^2,L)$be a two dimensional Finsler space with a Finslermetric function$L(x^1,x^2;y^1,y^2)$. We denote $\frac{\partial f}{\partial x^i}=f_i,\frac{\partial f}{\partial y^i}=f_{(x)}(i=1,2)$for any Finsler function $f(x^1,x^2;y^1,y^2)$. Here after, the suffices i,j run over 1, 2.

Since$L(x^1,x^2;y^1,y^2)$ is (1) *p*-homogeneous in (y^1,y^2)we have $L_{(j)(i)}y^i=0$which imply the existence of a function, so called the

Weierstrass invariant $W(x^1,x^2;y^1,y^2)$([4], [8])given by

(2.1)$\frac{L_{(1)(1)}}{(y^2)^2}=-\frac{L_{(1)(2)}}{y^1y^2}=\frac{L_{(2)(2)}}{(y^1)^2}=W(x^1,x^2;y^1,y^2)$

In a two-dimensional associated Riemannian space $R^2=(M^2,\alpha)$with respect to $L=\alpha$ and $\alpha^2-a_{ij}(x^1,x^2)y^iy^j$, the Weierstrass invariant $W_r$of $R^2$is written as

$$W_r=\frac{1}{\alpha^3}\{a_{11}a_{22}-(a_{12})^2\}.$$

Further L_j are still (l)p-homogeneous in(y^1,y^2), so that we get

(2.2) $$L_{j(i)}y^i=L_j$$

The geodesic equations in F^2 along curve $C:x^i=x^i(t)$are given by [1]

(2.3) $$L_i-\frac{dL_{(i)}}{dt}=0$$

Substituting (2.2) in (2.3), we get

(2.4) $$L_{1(2)}-L_{2(1)}+(y^1\dot{y}^2-y^2\dot{y}^1)W=0$$

which is called the Weierstrass form of geodesic equation in F^2([8], [9]),where $\dot{y}^i=dy^i/dt$. For the metric function $L(x,y;\dot{x},\dot{y})$, (2.4) becomes to

(2.5) $$\frac{\partial^2L}{\partial\dot{y}\partial x}-\frac{\partial^2L}{\partial\dot{x}\partial y}+(\dot{x}\ddot{y}-\dot{y}\ddot{x})\frac{\partial^2L}{(\partial\dot{y})^2}=0$$

Let $\Gamma=(\gamma^i_{jk}(x^1,x^2))$be the Levi-Civita connection of the associated Riemannian spaceR^2. We introduce the linear Finsler connection $\Gamma=(\gamma^i_jk,\gamma^i_0j,0)$ and the *h*- and c-covariant differentiation in $\Gamma^*$are denoted by $(;i,\ (i))$respectively, where the index (0) means the contraction withy^i. Then we have $y^i_{;j}=0,\alpha_{;i}=0$ and $\alpha_{(i);j}=0$.

On other hand, from (3.1) we have

(3.4) $$L_{(j);i} = L_{\alpha\beta}\,\beta_{;i}\alpha_{(j)} + L_{\beta\beta}\,\beta_{;i}b_j + L_\beta\beta_{;i}b_{j;i}.$$

Similarly to the case of $L(x^1x^2; y^1y^2)$ and $\alpha(x^1, x^2)$, we get the Weierstrass invariant$w(\alpha, \beta)$ as follows:

(3.5) $$w = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2} \;.$$

Substituting (3.5) in (3.4), we have

(3.6) $$L_{(j);i} = \alpha w\beta_{;i}(\alpha b_j - \beta\alpha_{(j)}) + L_\beta b_{j;i}\;.$$

From (3.3) and (3.6) we have

(3.7) $$L_{1(2)} - L_{2(1)} = \alpha w\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\} - L_\beta(b_{1;2} - L_{2;1})$$
$$+(y^1\gamma_{0\,0}^2 - y^2\gamma_{0\,0}^1)W.$$

If we put $y^i_{;0} = \dot{y}^i + \gamma_{0\,0}^i$, we get

(3.8) $$y^1\dot{y}^2 - y^2\dot{y}^1 = y^1y^2_{;0} - y^2y^1_{;0} - (y^1\gamma_{0\,0}^2 - y^2\gamma_{0\,0}^1).$$

Substituting (3.7) and (3.8) in (2.4), we have

(3.9) $$\alpha w\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\} - L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) + (y^1y^2_{;0} - y^2y^1_{;0})W\;,$$

where$\beta_{;i} = b_{r;i}y^r$. According to §2 of [6] , the relation of W, W_r and w is written as follows:

(3.10) $$W = (L_\alpha + \alpha w\gamma^2)W_r \quad ,$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$ and $b^2 = a^{ij}\,b_i b_j$.

Therefore (3.9) is expressed as follows:

(3.11) $$(L_\alpha + \alpha w\gamma^2)(y^1y^2_{;0} - y^2y^1_{;0})W_r - L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right)$$
$$+\alpha w\{b_{0;1}(\alpha b_2 - \beta\alpha_{(2)}) - b_{0;2}(\alpha b_1 - \beta\alpha_{(1)})\} = 0\;.$$

Thus we have the following

Theorem 3.1 In a two-dimensional Finsler space F^2 with an (α,β) – metric, the differential equation of a geodesic is given by (3.11).

Suppose that α be positive – definite. Then we may refer to an isothermal coordinate system $(x^i) = (x, y)$([5]) such that

$$\alpha = aE \;\;,\; a = a(x,y) > 0,\; E = \sqrt{\dot{x}^2 + \dot{y}^2} \;\;,$$

that is , $a_{11} = a_{22} = a^2, a_{12} = 0$ and $(y^1, y^2) = (\dot{x}, \dot{y})$.From $\alpha^2 = a_{ij}(x)y^iy^j$ we get $\alpha\alpha_{(i)(j)} = a_{ij} - a_{ir}a_{js}y^ry^s/\alpha^2$. Therefore we have$\alpha\alpha_{(1)(1)} = (a\dot{y}/E)^2$. and $W_r = a/E^3$ Furthermore the Christoffel symbols are given by

$$-E^3L_\beta(b_{1y}-b_{2x})-E^3a^2w(b_1\dot{y}-b_2\dot{x})b_{0;0}=0$$

where

(3.17) $$b_{0;0}=b_r s y^r y^s=(b_{1x}\dot{x}+b_{1y}\dot{y})\dot{x}+(b_{2x}\dot{x}+b_{2y}\dot{y})$$
$$+\frac{1}{a}\{(\dot{x}^2+\dot{y}^2)(a_x b_1+a_y b_2)-2(b_1\dot{x}+b_2\dot{y})(a_x\dot{x}+a_y\dot{y})\}$$

where $b_{ix}=\frac{\partial b_i}{\partial x}$, *and* $b_{iy}=\frac{\partial b_i}{\partial y}$ thus we have the following

Theorem 3.2 : In a two dimensional Finsler space F^2 with an (α,β)- metric, if we refer to an isothermal coordinate system (x,y)such that $\alpha=aE$, then the differential equation of a geodesic is given by (3.16) and (3.17).

4. Equation of Geodesics in a two dimensional Finsler with special(α,β)-metric $L=\alpha+\beta+\frac{\beta^2}{\alpha-\beta}$.

The (α,β) -metric $L(\alpha,\beta)=\alpha+\beta+\frac{\beta^2}{\alpha-\beta}$is called special$(\alpha,\beta)$metric .

(4.1) $$\begin{cases} L_\alpha=1-\frac{\beta^2}{(\alpha-\beta)^2}, L_{\alpha\alpha}=\frac{2\beta^2}{(\alpha-\beta)^3}, L_{\alpha\beta}=-\frac{2\alpha\beta}{(\alpha-\beta)^3}, L_{\beta\beta}=\frac{2\alpha^2}{(\alpha-\beta)^3} \\ w=\frac{L_{\alpha\alpha}}{\beta^2}=-\frac{L_{\alpha\beta}}{\alpha\beta}=\frac{L_{\beta\beta}}{\alpha^2}=\frac{2}{(\alpha-\beta)^3} \end{cases}$$

Substituting (4.1) in (3.16), we obtain the differential equation of a geodesic in an isothermal coordinate system (x,y) with respect to αas follows:

(4.2) $$\{\alpha(\alpha-\beta)(\alpha-2\beta)+2\alpha(b_1\dot{y}-b_2\dot{x})^2\}\{a(\dot{x}\ddot{y}-\dot{y}\ddot{x})+E^2(a_x\dot{y}-a_y\dot{x})\}$$
$$-E^3\alpha^2(\alpha-\beta)(b_{1y}-b_{2x})-2E^3a^2(b_1\dot{y}-b_2\dot{x})b_{0;0}=0$$

If the particular t of curve C is chosen x of(x,y) , then $\dot{x}=1, \dot{y}=y', \ddot{x}=0, \ddot{y}=y'', \sqrt{1+(y')^2}$.

(4.3) $$\{\alpha(\alpha-\beta)(\alpha-2\beta)+2\alpha(b_1y'-b_2)^2\}\{ay''+(1+(y')^2)(a_xy'-a_y)\}$$
$$-a(1+(y')^2)\{(1+(y')^2)\alpha(\alpha-\beta)(b_{1y}-b_{2x})-2\alpha(b_1y'-b_2)b^*_{0;0}\}=0$$

(4.4) $$b^*_{0;0}=(b_{1x}+b_{1y}y')+(b_{2x}+b_{2y}y')y'$$
$$+\frac{1}{a}\{\{1+(y')^2\}(a_xb_1+a_yb_2)-2(b_1+b_2y')(a_x+a_yy')\}$$

It seems quite complicated from, but y'' is given as a fractional expression in y' .

Thus we have the following

Theorem 4.1 Let F^2 be two-dimensional space with special Finsler metric. If we refer to a local coordinate system (x,y) with respect to α , then the differential equation of a geodesic $y=y(x)$ *of* F^2 is of the form

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